Theory of Types for Sums of Normal-Play Games

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Outline

Definition of Games

Definition (Combinatorial Games)

A *combinatorial game* is a two-player game which consists of

1. a set of positions to be played on/with,
2. a moving rule for both Louise (Left player) and Richard (Right player) when at a specific position,
3. a win rule which is a set of positions we designate as "terminating", such that, when reached, indicates that one of the players won the game.

Normal-Play Games

Definition (Normal-Play Games)

A *normal-play game* is a combinatorial game where a player wins if they are the last person with an available move.

Theorem

Normal-play games do not end in draws.

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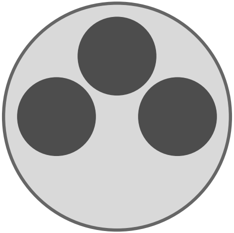
Normal-Play Pick Up Bricks

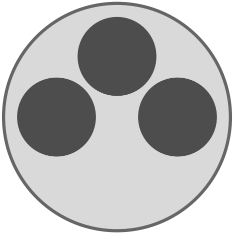
- Two Players
- Adjustable number of bricks
- Take 1 or take 2 bricks
- Player to take last bricks/Last turn wins

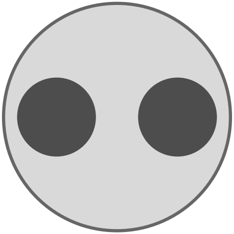


- Pick Up Bricks game with 3 bricks in position

Positions of games

Definition (Positions)





Example of two positions, A and B respectively. Demonstration above is of two different possible positions in the normal-play game *Pick Up Bricks*.

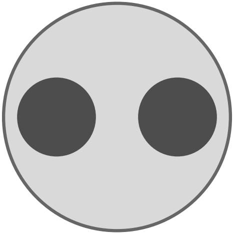
Types of Normal-Play Games

Definition (Types)

- *L type* means Louise has a winning strategy
- *R type* means Richard has a winning strategy
- *N type* means that the First/Next player in the game has a winning strategy
- *P type* means that the Second/Previous player in the game has a winning strategy

Type N Explanation

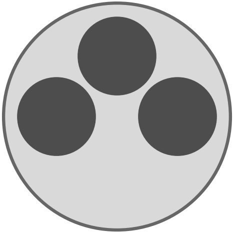
Using one of the Pick up bricks examples, we can see a position which is a type N:



- Player to take last turn/brick wins
- Players can either take 1 or 2 bricks
- If first player takes 2, then there will be no bricks remaining/no turns left
- First player has a was to always win

Type P Explanation

Using other Pick up bricks example, we can see a position which is type P:



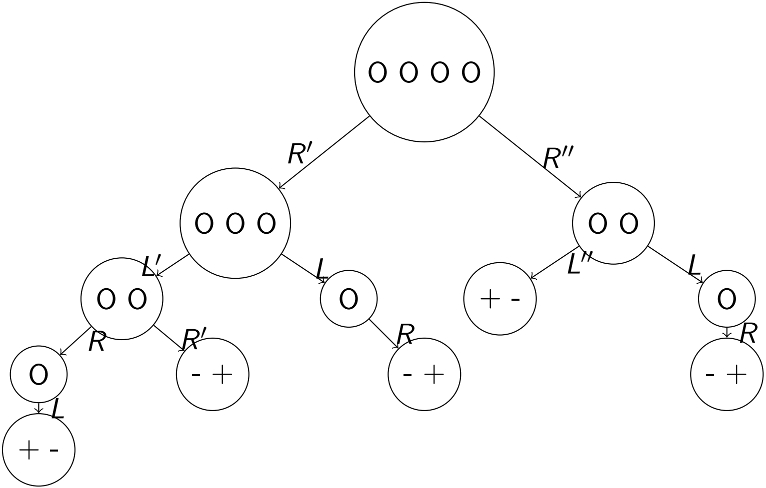
- Player to take last turn/brick wins
- Players can either take 1 or 2 bricks
- Regardless of first players choice, second player always gets last turn/brick meaning second player has a way to always win

Game Trees

Definition

Game Trees are used to organize all possible moves taken from a specific game position in a way that can eventually show all terminating moves.

Game Tree Example

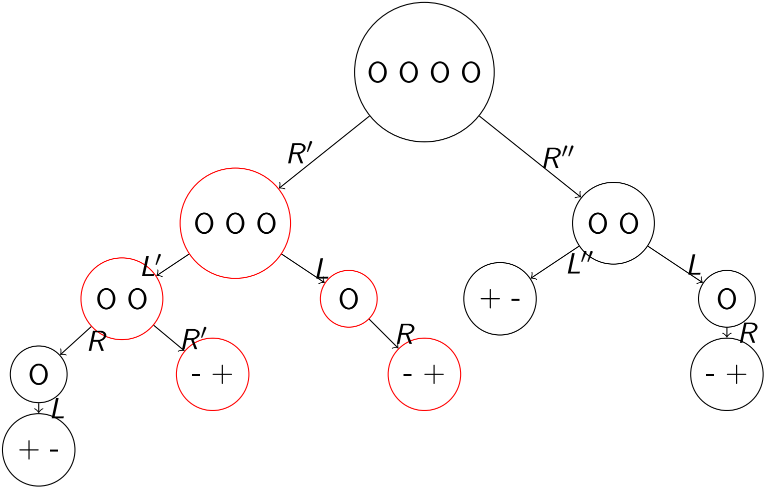


Terminating Node Notation

Our game tree holds terminating nodes notated as follows:

- A node holding " $+$ - " describes a winning outcome for **L**ouise as the plus is on the **L**eft side of the node.
- A node holding " - $+$ " describes a winning outcome for **R**ichard as the plus is on the **R**ight side of the node.
- A node holding " 0 0 " describes an outcome of a drawing strategy between both Richard and Louise as no one is gaining anything; we will not find this in the game tree of any normal-play games.

Winning Strategy



Sums of Positions

- Every normal-play positions has a type
- Positions can be added together
- The result, known as the sum, is influenced by the positions types

Sums of Positions Table

X	L	R	N	P
L	L	?	?	L
R	?	R	?	R
N	?	?	?	N
P	L	R	N	P

- Shows all possible sums of types when added together
- "?" indicates the sum is undetermined/undefinable

Sums of Positions Explained

If two positions have the same type, then their sum will also be that same type. This definition does not apply for type N positions.

$$L + L = L \tag{1}$$

$$R + R = R \tag{2}$$

$$P + P = P \tag{3}$$

$$N + N = ? \tag{4}$$

Sums of Positions Explained Same type

For example, L + L = L

- L wins in position A
- L wins in Position B
- All positions shown have a winning strategy for L, therefore, it can be discerned that player L will be the winner in both cases

Sums Of Positions Explained Type P

Type P interacts with other types in a peculiar way. This is because when a position with type P is added with the type of a different position, the sum will always be that of the other position.

Type P Sum Example

- Right position(A) = Type N
- Left position(B) = Type P

When added together, the sum of these two positions will be N.

Type P Sum Explained

This is because to win in type N, you must be the first player to make a move in that position. The first player to move will eventually win due to having the last turn in said position.

Due to making the last move in position B, their next move must be done in position A since it is the only position left. Since they just had their turn, this means that for position A they must move second. As we know, type P means the second player to make a move wins, therefore, the winner of type N will also be the winner of type P due to having to play second in a position in which the second player has a winning strategy.

Domineering Introduction

Now that we’ve gone through types, let’s use Domnineering to address the “?” chart spaces which illustrate that, by knowing the two initial types, we cannot determine the type of the sum of said games.

- Domineering: A normal-play game with a set of positions consisting of multiple rectangular arrays fused together. Whoever is the last to remove a 2×1 or 1×2 domino (over two unoccupied unit squares of the figure) wins, as Richard can eliminate 1×2 dominoes while Louise can only remove 2×1 dominoes.

Explanation of ?: Example A

An example of where a “?” is on our chart includes the sum of a type N and type R game of Domineering. The first example of a sum of games includes a 2×2 and 1×2 position of Domineering.

- $(2 \times 2) + (1 \times 2)$: Type N + Type R = Type N

Explanation of ?: Example B

Our second example also includes the sum between a type N and type R game of Domineering including a 2×2 position summed with a 1×4 position. Let's see what could be different!

- $(2 \times 2) + (1 \times 4)$: Type N + Type R = Type R

Acknowledgements and Citations

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Citations:

M. DeVos and D. A. Kent. *Game Theory: A Playful Introduction*. *American Mathematical Society* Vol. 80, 2016.